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Motion and Diffusion of a Passive Scalar in a Two-Dimensional Fluid

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The motion of a particle in two dimensions in a fluid is considered. The fluid flow is given and time independent. The complex fluid velocity potential can be viewed as a conformal transformation and after rescaling the time, the motion of the particle is uniform and rectilinear in the absence of diffusion. When diffusion of the particle also takes place the same ideas lead to a useful self-consistent approximation based on the average motion of the particle.

KEY WORDS: Diffusion in two-dimensional fluid; conformal transformation in fluids.

1. INTRODUCTION

We consider the motion of a particle in two dimensions suspended in a fluid where the fluid flow is given and time independent. The presence of the particle does not affect the flow. The particle will tend to move with the local fluid velocity, but it may also be subject to a random force of the Brownian type so that diffusion of the particle can also take place. The particle could also be considered to be a vortex which is free to move at the local fluid velocity. The fluid flow is determined by a complex velocity potential $W(z)$ $(z = x + iy)$ and we will be interested in the cases where the flow is due to vortices or sources and sinks when

$$\begin{aligned} W(z) &= \frac{-i}{2\pi} \sum_i \kappa_i \ln(z - z_i) \qquad \text{vortices} \\ &= \sum_i m_i \ln(z - z_i) \qquad \text{sources and sinks} \end{aligned} \tag{1}$$

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where z_i is the position of the ith vortex or source and sink, κ_i is the circulation, and m_i is the strength of a source or a sink ($m_i > 0$ for a source). Other forms of W are also possible, but these seem to be the most interesting situations. The complex fluid velocity $v_- = v_x - iv_y = \partial W/\partial z = W'$.

In this paper we use the fact that the equations of motion of the particle can be written in Hamiltonian form. This was pointed out a long time ago by Lamb[1] and Lin[2] and also used by Onsager.[3] These latter two papers introduced us to the subject. This problem is also being studied by Koplik and Redner[4] who suggested some of the examples used here.

2. MOTION IN THE ABSENCE OF DIFFUSION

We first consider the motion of the particle when there is no random Brownian motion so that the particle moves with the local fluid velocity and its equation of motion is

$$\frac{dz}{dt} = W'^* \tag{2}$$

The quantity $-i(W - W^*)$ is the stream function and is conserved:

$$\frac{d}{dt}(W - W^*) = W' \frac{dz}{dt} - W'^* \frac{dz^*}{dt} = 0 \tag{3}$$

We can view the relation $W = W(z)$ for the complex velocity potential as a conformal transformation from the z to the W plane. In the W plane the flow is uniform and constant along the real axis and the particle will move in a straight line at constant velocity. It is clear that time will proceed differently for the z particle and W particle. We call the time scale for the W particle T and show below that the relation between T and t is

$$\frac{dT}{dt} = |W'|^2 = |\mathbf{v}|^2 \tag{4}$$

where the right-hand side depends on the trajectory of the particle.

This relation has been introduced previously in the conformal representation of the classical mechanics of a particle.[5] We now have

$$\frac{dW}{dt} = \frac{dW}{dT}\frac{dT}{dt} = W' \frac{dz}{dt} = |W'|^2 \tag{5}$$

Thus

$$\frac{dW}{dT}=1, \qquad W=T+W_0 \tag{6}$$

where W_0 is the initial value of W at $T=t=0$. In the W space the particle has constant velocity of unity along the real axis. By inverting the relation $W=W(z)$ we can now express the right-hand side of (4) as a function of T and this equation can be integrated to give the relation between T and t with the initial condition $T=t=0$. The motion of the particle is then determined by (6).

In the case where the flow is produced by vortices of circulation κ_i we have the integral

$$\int_C W'\,dz=\int_0^{T_0} W'\frac{dz}{dt}\frac{dt}{dT}dT=\int_0^{T_0} dT=\sum_{i \text{ in } C} \pm\kappa_i \tag{7}$$

where C is a closed trajectory of the particle and the sum is over the circulations of the vortices enclosed by the trajectory with the signs determined by the sense of the contour. Thus the period of the motion in the scaled time is $T_0=\sum_{i \text{ in } C} \pm\kappa_i$ and is determined by the total circulation of the enclosed vortices. The area enclosed by the trajectory is

$$A=\frac{1}{2i}\int_C z^*\,dz=\frac{1}{2i}\int_0^{T_0}\frac{z^*}{W'}\,dT \tag{8}$$

Similarly the length L of the trajectory can be shown to be

$$L=\int_0^{T_0}\frac{dT}{|W'|} \tag{9}$$

Now consider a trajectory which ends up at a sink. The integral $\int_C W'\,dz$ along such a trajectory will diverge, showing that the scaled time to reach the sink is infinite.

Thus the motion of the particle is very simple in terms of the scaled time T and it is only necessary to integrate (4) to relate T to the real time t. We consider some examples.

(a) Two equal vortices $W=(-i\kappa/2\pi)\ln(z^2-a^2)$. We have two vortices located at $z=\pm a$. Then, using (6), we have

$$z^2=a^2+e^{2\pi iW/\kappa}=a^2(1+pe^{2\pi iT/\kappa}) \tag{10}$$

where $p=(x_0^2-a^2)/a^2$ and x_0 is the initial position of the particle, which, without loss of generality, we take on the real axis with $|x_0|>a$. This expression shows that for $p^2 < 1$ the orbit is periodic in T with period κ and remains close to one or the other vortex. For $p^2 > 1$ the orbit encompasses both vortices and the period is 2κ.

To relate the time scales, we have

$$W' = \frac{-i\kappa}{\pi} \frac{z}{z^2 - a^2} = \frac{-i\kappa}{\pi} e^{-2\pi i W/\kappa}(a^2 + e^{2\pi i W/\kappa})^{1/2} \tag{11}$$

and using (6), Eq. (4) becomes

$$\frac{dT}{dt} = \left(\frac{\kappa}{\pi a p}\right)^2 \left(1 + p^2 + 2p \cos \frac{2\pi T}{\kappa}\right)^{1/2} \tag{12}$$

The time scales are then related by

$$t = \frac{\pi a^2 p^2}{\kappa(1+p)} \int_0^{\pi T/\kappa} \frac{d\alpha}{(1 - k^2 \sin^2 \alpha)^{1/2}} = \frac{\pi a^2 p^2}{\kappa(1+p)} F\left(\frac{2\pi T}{\kappa}, k\right) \tag{13}$$

where $k^2 = 4p/(1+p)^2$ and F is the elliptic integral of the first kind.

If $p^2 < 1$, the motion remains close to one or the other vortex and the period is $T_0 = \kappa$. In real time from (13) the period is

$$t_0 = \frac{2\pi a^2 p^2}{\kappa(1+p)} K(k) \tag{14}$$

where K is the first complete elliptic integral. The area enclosed by the orbit from (8) is

$$A = \frac{a^2}{2(1+p)^2} [(1+p)^2 E(k) - (1-p^2) K(k)] \tag{15}$$

where E is the complete elliptic integral of the second kind. The length of the trajectory from (9) is

$$L = \pi a p F(\tfrac{1}{4}, \tfrac{1}{4}, 1, p^2) \tag{16}$$

where F is a hypergeometric function.

If $p^2 > 1$, the motion encompasses both vortices, the period is $2t_0$, the area of the orbit is $2A$, and the length is $2L$. For $p = 1$ the particle is on the separatrix and the time to reach the origin (stagnation point) is infinite.

(b) Ring of n vortices $W = (-i\kappa/2\pi) \ln(z^n - a^n)$. The previous results are easily extended to the case of n vortices in a ring of radius a. We have

$$z^n - a^n = (x_0^n - a^n)\, e^{2\pi i T/\kappa} \tag{17}$$

where x_0 is the initial position of the particle with $x_0 > a$. The time scales are related by

$$t = \frac{4\pi a^2 p^2}{n^2 \kappa (1+p)^{2(n-1)/n}} \int_0^{\pi T/\kappa} \frac{d\alpha}{(1-k^2 \sin^2 \alpha)^{(n-1)/n}} \tag{18}$$

where $p = (x_0^n - a^n)/a^n$ and $k^2 = 4p/(1+p)^2$. Again the motion is confined to the vicinity of one vortex if $p^2 < 1$ and the period is $T_0 = \kappa$ or

$$t_0 = \frac{4\pi^2 a^2 p^2}{n^2 \kappa (1+p)^{2(n-1)/n}} F\left(\frac{n-1}{n}, \frac{1}{2}, 1, k^2\right) \tag{19}$$

where F is a hypergeometric function. If $p^2 > 1$, the motion extends around the ring and the period is nt_0. Other properties may be obtained as in example (a).

(c) Two vortices of opposite circulation $W = (-i\kappa/2\pi)\ln((z-a)/(z+a))$. We have two vortices of opposite circulation located at $\pm a$. The motion is always confined to the vicinity of one or the other vortex and

$$z = a \frac{(x_0 + a) + (x_0 - a) e^{2\pi i T/\kappa}}{(x_0 + a) - (x_0 - a) e^{2\pi i T/\kappa}} \tag{20}$$

where x_0 is the initial position of the particle on the real axis. The time scales are related by

$$t = \frac{4\pi a^2 p^2}{\kappa (1-p^2)^{3/2}} \left[\tan^{-1} \alpha(T) - \frac{p\alpha(T)}{1+\alpha^2(T)} \right] \tag{21}$$

where $\alpha(T) = [(1-p)/(1+p)]^{1/2} \tan(\pi T/\kappa)$ and $p = (x_0^2 - a^2)/(x_0^2 + a^2)$.

The period is $T_0 = \kappa$ or

$$t_0 = \frac{4\pi^2 a^2 p^2}{\kappa (1-p^2)^{3/2}} \tag{22}$$

(d) Line of vortices $W = (-i\kappa/2\pi)\ln \sin(\pi z/a)$. We have a line of identical vortices at $z = na$, where n is an integer. Then from (6)

$$\sin \frac{\pi z}{a} = i \sinh \frac{\pi y_0}{a} e^{2\pi i T/\kappa} \tag{23}$$

For simplicity we assume the particle starts on the imaginary axis at y_0 and set $s_i = \sinh(\pi y_0/a)$. Then from (4)

$$t = \frac{2a^2 s_i^4}{\pi \kappa (1+s_i^2)} \int_0^{2\pi T/\kappa} \frac{d\alpha}{(1-k_i^2 \sin^2 \alpha)^{1/2}} = \frac{2a^2 s_i^4}{\pi \kappa (1+s_i^2)} F\left(\frac{2\pi T}{\kappa}, k_i\right) \tag{24}$$

where $k_i^2 = 4s_i^2/(1+s_i^2)^2$. For $s_i^2 < 1$ the motion is confined to the vicinity of a single vortex and the period is

$$t_0 = \frac{8a^2 s_i^4}{\pi\kappa(1+s_i^2)} K(k_i) \tag{25}$$

For $s_i^2 > 1$ the motion is along the line and is periodic with period t_0.

(e) Sink at the origin $W = -m \log z$. This problem is of course trivial and the particle ends up at the sink in a finite time. It illustrates a new feature of the time scale T. In terms of T

$$z = e^{-W/m} = x_0 e^{-T/m} \tag{26}$$

and as $T \to \infty$ the particle reaches the origin from the initial point x_0. Equation (4) is

$$\frac{dT}{dt} = \frac{m^2}{x_0^2} e^{2T/m} \tag{27}$$

which is integrated to give

$$z = x_0 \left(1 - \frac{2m}{x_0^2} t\right)^{1/2} \tag{28}$$

(f) Source and sink $W = m \log[(z+a)/(z-a)]$. We have an equal sink and source at $\pm a$, respectively. Solving for z and using (6), we have

$$z = a \frac{(z_0 + a)e^{T/m} + z_0 - a}{(z_0 + a)e^{T/m} - (z_0 - a)} \tag{29}$$

where z_0 is the initial position of the particle. As $T \to \infty$ the particle ends up at the sink. The real time is determined by

$$t = \frac{4a^2}{m} \int_0^{T/m} \frac{dx}{F(x)} \tag{30}$$

where

$$F(x) = 4 + pq^* + p^*q - 2(p+p^*)e^x - 2(q+q^*)e^{-x} + |p|^2 e^{2x} + |q|^2 e^{-2x} \tag{31}$$

and $p = q^{-1} = (z_0 + a)/(z_0 - a)$. The time for the particle to reach the sink starting at z_0 is

$$t_0 = \frac{4a^2}{m} \int_0^\infty \frac{dx}{F(x)} \tag{32}$$

These examples should be sufficient to illustrate the power and utility of this method.

3. MOTION WITH DIFFUSION

In this case the particle moves with the fluid and is also subject to a random force $f(t)$ and (2) is replaced by

$$\frac{dz}{dt} = W^{*\prime} + f^*(t) \tag{33}$$

where f has the correlation function

$$\langle f(t)\, f^*(t') \rangle = 4D\delta(t - t') \tag{34}$$

It is more convenient to use the equivalent Fokker–Planck equation for the distribution $P(z, z^*, t)$ which satisfies

$$\frac{\partial P}{\partial t} + \frac{\partial}{\partial z} W^{*\prime} P + \frac{\partial}{\partial z^*} W' P = 4D \frac{\partial^2}{\partial z\, \partial z^*} P \tag{35}$$

This equation, except in some rather trivial cases, is not easily solvable. Our previous considerations lead to a useful self-consistent approximation. Introducing the coordinate $W = W(z)$ in (35), we get

$$\frac{\partial P}{\partial t} + |W'|^2 \left(\frac{\partial}{\partial W} + \frac{\partial}{\partial W^*} - 4D \frac{\partial^2}{\partial W\, \partial W^*} \right) P = 0 \tag{36}$$

In the stationary situation ($\partial P/\partial t = 0$) the problem in the W space is simple and requires the solution of a problem with constant flow along the real axis. It is ($W = u + iv$)

$$\left[\frac{\partial}{\partial u} - D \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \right] P = 0 \tag{37}$$

Returning to the time-dependent case, we let

$$\langle |W'|^2 \rangle = \int P(z, z^*, t)\, |W'|^2\, dz\, dz^* \tag{38}$$

which is the average square drift velocity of the particle. We now introduce a new time scale according to the average motion of the particle

$$\frac{dT}{dt} = \langle |W'|^2 \rangle \tag{39}$$

When this is introduced into (36) we get

$$\frac{\partial P}{\partial T} + \frac{|W'|^2}{\langle |W'|^2 \rangle} \left(\frac{\partial}{\partial W} + \frac{\partial}{\partial W^*} - 4D \frac{\partial^2}{\partial W \, \partial W^*} \right) P = 0$$

We now neglect fluctuations in the square of the drift velocity and the above equation becomes

$$\frac{\partial P}{\partial T} + \left(\frac{\partial}{\partial W} + \frac{\partial}{\partial W^*} - 4D \frac{\partial^2}{\partial W \, \partial W^*} \right) P = 0 \tag{40}$$

This is the ordinary diffusion equation with solution (in infinite space)

$$P(W, W^*, T) = \frac{1}{4\pi DT} \exp\left(-\frac{1}{4DT} |W - W_0 - T|^2 \right) \tag{41}$$

where W_0 is the initial value of W at $T = t = 0$. It remains to integrate (38), where

$$\langle |W'|^2 \rangle = \int P(W, W^*, T) \, |W'|^2 \, dW \, dW^* \tag{42}$$

is a function of T. Again the motion is very simple in terms of the scaled time T.

An estimate of the neglected terms in (40) requires that $Dt/l^2 = R^{-1} < 1$, where l is the length scale associated with the flow. Thus (40) is exact in the absence of diffusion and accurate for short times in the absence of fluid motion. R plays the role of the Reynolds number in this problem. As an example we consider the case of a source and a sink

Source and sink $W = -m \log((z-a)/(z+a))$. This transformation maps the z plane into the infinite strip $-\pi m < v < \pi m$ in the $W = u + iv$ plane. The source and sink are at $u = \mp \infty$, respectively. Suppose particles are flowing out of the source at a uniform rate $2\pi m P_0$, so that we have a steady state. The only solution of (37) periodic in v and not diverging at $u = \pm \infty$ is $P = P_0$ a constant. Thus in this steady state the concentration of particles is everywhere constant. The complex particle current is

$$J_+ = J_x + iJ_y = P_0 W'^* = P_0 m \left(\frac{1}{z^* + a} - \frac{1}{z^* - a} \right) \tag{43}$$

We now consider the time-dependent problem where the particle at $t=0$ is at z_0. In the W plane $u_0+iv_0=-m\log((z_0-a)/(z_0+a))$. The appropriate normalized solution of (40) which is periodic along the v axis is

$$P(W, W^*, T)=\frac{1}{(16\pi^3 DTm^2)^{1/2}}\sum_{n=-\infty}^{\infty} e^{(in(v-v_0)/m)-(DTn^2/m^2)}e^{-(u-u_0-T)^2/4DT} \tag{44}$$

where n is an integer. Using this distribution, it is easy to integrate (42) and the relation between the time scales (39) becomes

$$t=\frac{4a^2}{m^2}\int_0^{T/m}\frac{dx}{F_1(x)} \tag{45}$$

where

$$F_1(x)=4+(pq^*+p^*q)e^{-2\alpha x}-2(p+p^*)e^x-2(q+q^*)e^{-x} \\ +|p|^2 e^{2x(1+\alpha)}+|q|^2 e^{-2x(1-\alpha)} \tag{46}$$

$\alpha=2D/m$ and p and q have been given below (31). As $T\to\infty$ the particle reaches the sink, so the average time to reach the sink is

$$t_{av}=\frac{4a^2}{m^2}\int_0^{\infty}\frac{dx}{F_1(x)} \tag{47}$$

The particle distribution in the z space is

$$P(z, z^*, t)=P(W, W^*, T)\,|W'(z)|^2 \tag{48}$$

4. OTHER TRANSFORMATIONS

In Section 2 we chose to map the flow into the motion of a particle in a uniform flow along the real axis. Other possible mappings are also possible. Suppose we have found the motion of the particle in a given flow described by a complex velocity potential $W(z)$. We may now make a conformal transformation $z=f(\zeta)$ to get a different flow described by the complex velocity potential $U(\zeta)=W(f(\zeta))$. From (2) we have

$$\frac{dz}{dt}=\frac{d\zeta}{dt}f'=\frac{dW^*}{dz^*}=\frac{1}{f'^*}\frac{dU^*}{d\zeta^*} \tag{49}$$

We now introduce a new time scale τ for the ζ particle determined by

$$\frac{dt}{d\tau} = |f'|^2 \tag{50}$$

From (49)

$$\frac{d\zeta}{d\tau} = \frac{dU^*}{d\zeta^*} \tag{51}$$

and the motion of the ζ particle in the new flow is described by an equation of the same form as (2).

5. CONCLUSIONS

The complex velocity potential $W = W(z)$ can be viewed as a conformal transformation. A particle which obeys (2) executes the very simple uniform rectilinear motion $W = T + W_0$ in the W space in terms of the new time T. This leads to a simple and powerful method for integrating equations of motion of the form (2). Simple solutions are obtained in the case where the relation $W = W(z)$ can be explicitly inverted. We mention some other simple cases which are easily integrable. (a) $W = (-i\kappa/2\pi) \ln[(z^2 - a^2)/z^n]$, with $n = 1$ or 2 corresponds to two equal vortices at $\pm a$ and a negative vortex at the origin of strength κn. (b) $W = (-i\kappa/2\pi) \ln((z^n - 1)/(z^n + 1))$ corresponds to a ring of $2n$ vortices of alternating sign. (c) $W = (-i\kappa/2\pi) \ln \tan(\pi z/a)$ corresponds to a line of vortices of alternating sign.

What we learn from these examples is that small periodic orbits are the most common and that generally large orbits require special conditions. Thus, mixing of the passive scalar will not readily occur unless large-scale fluid flow and diffusion are both present. When diffusion of the particle also takes place the same ideas lead to a simple and useful self-consistent approximation based on the average motion of the particle.

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